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Introduction of Complex Laplacian to Multi-Agent Systems

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Abstract-The paper concentrates on the fundamental coordination problem that requires a network of agents to achieve a specific but arbitrary formation shape. A new technique based on complex Laplacian is introduced to address the problems of which formation shapes specified by inter-agent relative positions can be formed and how they can be achieved with distributed control ensuring global stability. Concerning the first question, we show that all similar formations subject to only shape constraints are those that lie in the null space of a complex Laplacian satisfying certain rank condition and that a formation shape can be realized almost surely if and only if the graph modeling the inter-agent specification of the formation shape is 2-rooted. Concerning the second question, a distributed and linear control law is developed based on the complex Laplacian specifying the target formation shape, and provable existence conditions of stabilizing gains to assign the eigenvalues of the closed-loop system at desired locations are given. Moreover, we show how the formation shape control law is extended to achieve a rigid formation if a subset of knowledgable agents knowing the desired formation size scales the formation while the rest agents do not need to re-design and change their control laws.

Index Terms—Distributed control, formation, graph Laplacian, multi-agent systems, stability

I. INTRODUCTION

In recent years, there has been a tremendous surge of interest among researchers from various disciplines of engineering and science in a variety of problems on networked multi agent systems. Modelling the interaction topology of distributed Agents as a graph, a main stream of research ([3], [23], [28], [31], and [35]) concentrates on understanding and designing the mechanisms from the structure point of view on how collective behaviours emerge from local interaction in absence of high level centralized supervision and global information exchange. An interesting example and area of on-going research is the control of teams of autonomous mobile robots, unmanned aerial vehicles (UAVs), and autonomous underwater vehicles (AUVs), so that they work cooperatively to accomplish a common goal without centralized control and a global coordinate system. As teams of agents working together in formation can be found in various applications such as satellite formation flying, source seeking and exploration, ocean data retrieval, and map construction, much attention has been given to the control of formations. Studies concerning this subject focus primarily on the formation architecture as well as the stability of the formation systems. The former mainly concentrates on defining formation using graph-theoretic rigidity [4], [16], [17], [22],[33], [40], while the latter concerns stabilization to a formation[7], [8], [10], [24], [32], [38], [39] and control of formation shape in moving [3], [5], [12]-[14], [20].With regard to rigid formations, there have been several types of control strategies, e.g., affine feedback control laws[1], [2], [10], [18], [25], [29], [34], nonlinear gradient control laws [8], [13], [20], [24], [38], and very recently, angle-based control algorithms [6], [21], [30]. The goal is to achieve affirmation with a determined size, which has only freedoms of translations and rotations. On the other hand, [9] studies the formation control problem with the objective of steering a team of agents into a formation of variable size. By allowing the size of the formation to change, the group can dynamically adapt to changes in the environment such as unforeseen obstacles, adapt to changes in group objectives, or respond to threats. In this paper, we concentrate on the fundamental coordination problem that requires the agents to achieve a specific but arbitrary formation shape. By *formation shape*, we are referring to the geometrical information that remains when location, scale, and rotational effects are removed. Thus, formation shapes invariant under the Euclidean similarity transformations of translation, rotation and scaling. The formation shape control problem is of its own interest if the agents do not have notion of the world coordinate system's origin as well as unit of length or if the goal is to just form a pattern such that the agents can then agree on their respective roles in a subsequent, coordinated action. Moreover, formation shape control also serves as a basis for rigid formation control. As we show in this paper, when formation shape control is possible, a task of rigid formation control can be accomplished with a subset of knowledgeable agents knowing the desired formation size, for which the advantage is that the rest agents do not need to redesign and change their control laws in order to achieve the desired formation scaled by the desired size. In this context, the main research questions are which formation shape specified by inter-agent relative positions can be formed and how they can be achieved with distributed control ensuring global stability. Concerning the first question, we introduce the notion of similar formation and show that all similar formations subject to only shape constraints are those that lie in the null space of a complex Laplacian satisfying certain rank condition. Moreover, we prove that an equivalent graphical condition such that a formation shape can be realized is that the graph modelling the inter-agent specification of the formation shape is 2-rooted. This is a kind of new connectivity in graph theory, meaning that there exists a subset of two nodes from which every other node is 2-reachable. Concerning the second question, we develop a distributed and linear control law that is based on the complex Laplacian specifying the target formation shape and can be locally implemented byonboard sensing using relative position measurements. It is shown that for almost all complex Laplacian specifying the target shape, stabilizing gains exist to ensure not only globally asymptotic stability but also other performance specifications such as robustness and fast convergence speed by assigning the eigenvalues of the closed-loop system at desired locations. A procedure is also provided on how to find stabilizing gains. In addition, we show how the formation shape control law is extended to achieve a rigid formation with the formation size Controlled by at least a pair of agents when they know the desired formation size. The contributions of the paper are three-fold. First, the paper presents a systematic approach based on complex Laplacianfor the formation shape control problem that is significant in the field. The work is an extension of our conference paper [37], including new developments on systematic construction of complex Laplacian for a given target formation shape, on finding stabilizing gains arbitrarily assigning the eigenvalues of the closed-loop system, and on how a rigid formation can be accomplished by controlling a subset of agents while the remaining agents still implement the same formation shape control law. Second, it provides a new way for rigid formation control by imposing one edge length constraints. Compared with globally rigid formation specified by integrant distances and nonlinear gradient control laws, the approach requires much less relative position measurements. Also, the approach makes possible that a large number of agents achieve a rigid formation almost globally by combining the nonlinear gradient control laws for a small number of agents to attain the edge length constraints, which are well studied with ensured almost global stability properties ([7], [8], [15], [20], [38]), and the simple linear formation shape control laws for the remaining agents. The approach has an advantage that a group of agents can easily change their formation size without a re-design of the control laws for all the agents. This property is more desirable in situations where the environment change is only observed by a minority of agents in the group. Most importantly, duet the use of linear control laws by most agents, it brings the hope by extending the approach to solve those challenging formation control problems in the setup of directed (time varying) topology and in higher dimensional spaces. Third, the work provides an original analysis for understanding the relationship between complex graph Laplacian and graphical connectivity, which researchers from other disciplines may be interested in. Though the paper mainly focuses on the formation control problem of networked agents in the plane. The methods however, are general, and they have applicability beyond multi robot formations, e.g., distributed beam forming of communication systems and power networks where a pattern in the state is an objective. The organization of the paper is as followers. We review the notations and some knowledge of graph theory in Section Iain Section III necessary and sufficient (algebraic and graphical) conditions are analysed for similar formations. Global stabilization and stability analysis of multi-agent formations are presented in Section IV. Simulation and experiment results are given in Section V. Section VI concludes our work and points out several open problems along the path introduced in the paper.

II. NOTATION AND GRAPH THEORY

A. Notation

We denote by C and R the set of complex and real numbers, respectively. $\imath = \sqrt{-1}$denotes the imaginary unit. For acomplex number $p \in C, |p|$represents its modulus. For a set$\varepsilon, |\varepsilon|$, represents the cardinality. 1_n represents the *n*-dimensional

vector of ones and I_n denotes the identity matrix of order *n*. A *block diagonal matrix*, which has main diagonal block matrices$A_1, \ldots, A_n$and off- diagonal blocks zero matrices, is denoted

asbd [$A_1, \ldots, A_n$].

B. Graph Theory

An *undirected graph* $G = (V, \varepsilon)$consists of a non-empty node set *V*={1,2,..,n}and an edge set $\varepsilon \subseteq V \times V$where an edge of G is a pair of un-ordered nodes. Undirected graphs can beconsidered as a special class of directed graphs with theedges consisting of pairs of ordered nodes, called *bidirectional graph*, for which each edge is converted into two directed edges, *(i,j)* and *(j,i)* In what follows we use the notion of bidirectional graph (or simply a graph for short) because the graph model we study is topologically equivalent to an undirected graph but different weights are considered on the edges of different order for the same pair of nodes. However, the graphical representation of undirected graphs is still used throughout the paper (i.e., we draw a line rather than two lines with arrows in the graph as the edges). A *walk* in a graph G is an alternating sequence $p: v_1 e_1 v_2 e_2 \ldots e_{k-1} v_k$of nodes v_i and edges e_i such that $e_i = (v_i, v_{i+1})$for every $i = 1, 2, \ldots, k-1$.We say that $p$is a *walk* from $v1$to vk.If the nodes of a walk $p$are distinct, $p$ia a *path*. v_1 Andv_k are called *terminal nodes a*nd other nodes are called *internal nodes*. A path is called *Hamiltonian path* if it visits every node in the graph exactly once. Throughout the paper, we let $N_i = \{j : (j, i) \in \varepsilon\}$. In the paper, we assume that a bidirectional graph does not have self-loops, which means $i \notin N_i$ for any node *i*.Next, we introduce two concepts.

Definition 2.1: For a bidirectional graph$G$ .a node $v$is said to be *2-reachable* from a non-singleton set U of nodes if there exists a path from a node in U to v after removing any one node except node v

Definition 2.2: A bidirectional graph $G$is said to be *2-rooted* if there exists a subset of two nodes, from which every other node is 2-reachable. These two

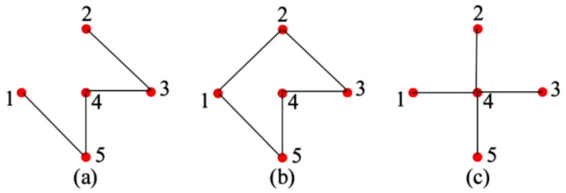

Fig. 1. Graphs that is 2-rooted and not 2-rooted

Nodes are called *roots* in the graph. Consider for example the graphs in Fig. 1. In Fig. 1(a), let U={1,2} and it can be checked that node 3 is 2-reachable from U as after removing any one other node we are still able to find a path from a node in U to node 3. Similarly, it is known that node 4 and 5 are also 2-reachable from U in Fig. 1(a). Thus the graph in Fig. 1(a) is 2-rooted with the two roots being nodes1 and 2. In Fig. 1(b), the graph is 2-rooted as well and any two nodes can be considered as roots in the graph. In Fig. 1(c), again let U={1,2 } and it is known that node 3 is not 2-reachable from the set U as if we remove node 4, there is no path anymore from any node in U to node 3. Furthermore, it can be verified that no matter how we select a subset of two nodes, there always exists another node that is not 2-reachable from the selected subset of nodes. Therefore, the graph in Fig. 1(c) is not 2-rooted.

Finally, we introduce a complex Laplacian for a bidirectional graph. The complex-valued Laplacian L of a bidirectional graph G is defined as follows: The ij^{th}entry

$$L(i,j) = \begin{cases} -w_i & if\ i \neq j\ and\ j \in N_i \\ 0 & if\ i \neq j\ and\ j \notin N_i \\ \sum_{j \in N_i} w_{i,j} & if\ i = j \end{cases}$$

Where$w_{ij} \in \mathbb{C}$ Note that the graph is a bidirectional graph, so the pattern of zero and nonzero entries of L is symmetric, but L may not be symmetric due to possibly different weights on the edges of the same pair of nodes but with different order.

The definition of complex Laplacian is nothing new from real Laplacian except that the nonzero entries can be complex numbers. Consequently, it is also true that a complex Laplacian has at least one eigenvalue at the origin that's associated

Eigen vector is 1_n (namely, $L1_n$=0).

A *permutation matrix* is a square binary matrix that has exactly one entry 1 in each row and each column and 0'selsewhere. Renumbering the nodes of a graph is equivalent to apply a permutation transformation to the Laplacian. That is, $L' = PLP^T$where Land L'are the Laplacian before and after renumbering the nodes, and P is the corresponding permutation matrix.

III. NECESSARY AND SUFFICIENT CONDITIONS FOR SIMILAR FORMATIONS

A. Overview of Rigid Frameworks With Distance Specifications

To introduce the notion of similar formation we will embed a graph in the complex plane$\mathbb{C}$ as as a framework. Let $G = (v, \varepsilon)$be a bidirectional graph with n nodes. We embed G into $\mathbb{C}$assigning to each node i a location (complex number) $\xi_i \in \mathbb{C}$

In a reference frameΣ. Define the n-dimensional composite complex vector $\xi = [\xi 1, \xi 2, \dots, \xi n]T \in Cn$

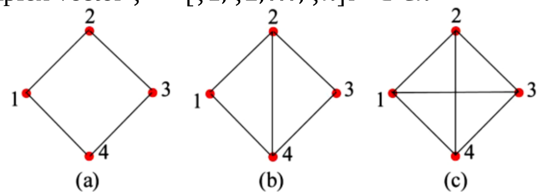

Fig. 2 (a) Not rigid. (b) Rigid but not globally rigid. (c) Globally rigid.

Formation configuration

in the reference frame Σ. A *framework* is a pair (G, ξ) Throughout the paper, we assume that $\xi_i \neq \xi_j$ and $i \neq j$meaningthat no two nodes are overlapping each other. In the following, we review a little bit about rigidity of graphs using the distance specifications. The materials below are taken from [24]. Associated with the framework (G, ξ), define a function $g: \mathbb{C}^n \to \mathbb{R}^{|\mathcal{E}|}$by

$$g(\xi) \coloneqq [\dots |\xi_i - \xi_j|^2 \dots .]^T$$

Called a *rigid function*. The kth component of (ξ), $|\xi_i - \xi_j|^2$, corresponds to the edge $e_k \epsilon\ \varepsilon$where nodes i and j are connected by $e_k$and specifies a desired edge length d_k. Let $d = [\dots . d_k \dots .]^T$be the composite vector describing thedistance specifications on the edges in G. Then the notions of rigidity and global rigidity can be stated as follows.

Definition 3.1: A framework (G, ξ)specified by $g(\xi) = d$is *rigid* if there exists a neighbourhood $\mathcal{B} \subset \mathbb{C}^n$ of ξsuch that

$$g^{-1}(d) \cap \mathcal{B} = \{c_1 1_n + e^{i\theta}\xi\colon\ c_1 \in\ \mathbb{C}\ and\ \theta \in [0,2\pi)\}$$

Definition 3.2: A framework (G, ξ)specified by $g(\xi) = d$is *globally rigid* if

$$g^{-1}(d) = \{c_1 1_n + e^{i\theta}\xi\colon\ c_1 \in\ \mathbb{C}\ and\ \theta \in [0,2\pi)\}$$

The level set $g^{-1}(d)$consists of all possible points that have the same edge lengths as the framework (G, ξ)The set $c_1 1_n + e^{i\theta}\xi\colon\ c_1 \in \mathbb{C}$ and $\theta \in [0,2\pi)\}$consists of points related byRotations $\theta$and translations c_1i.e., rigid body motions, of the framework (G, ξ)Therefore, a framework is rigid if thelevel set $g^{-1}(d)$in a neighbourhood of ξcontains only pointscorresponding to rotations and translations of the formationconfiguration ξ.A framework is globally rigid if the level set $g^{-1}(d)$ in $\mathbb{C}^n$contains only points corresponding torotationsand translations of the formation configuration ξFor example, consider the framework in Fig. 2(a). It is possible to translate only nodes 1 and 2, while maintaining the four edge lengths, to a formation that is not attained by rigid body motions, so the framework specified by $g(\xi) = d$is not rigid. If we add one more edge to obtain a framework as in Fig. 2(b), the only motion to maintain the five edge lengths in the neighbourhood is a rigid body motion (rotations and translations). As a result, the framework is rigid. But node 1can has a flip along the edge connecting 2 and 4, while the edge lengths are preserved, so it is not globally rigid. Fig. 2(c) shows a globally rigid framework.

B. Linear Constraints and Similar Formations

From the preceding subsection, it is clear that in order to make a framework rigid (or globally rigid), each node in the graph has to have at least two neighbours as otherwise if a node has only one neighbour, this node can swing around its neighbour.

By observing this fact, we will then introduce a new linear constraint for a framework rather than the distance constraints on the edges of the graph. For each node *i* in the graph, since it has at least two neighbours, we can define a linear constraint for the framework as follows:

$$\sum_{j\in N^i} w_{ij}\left(\xi_i - \xi_j\right) = 0$$

For appropriate complex weights w_{ij}defined on the edges linking to node *i*. The complex weights ake the relative state vectors rotated and scaled so that the summation becomes 0 for a given framework, and thus provide a linear constraint. Take Fig. 3 as an example. Node 3 has two neighbours (namely, 2 and 4). So the complex weights w_{32} and w_{34}rotate and scale the relative states $\xi_2 - \xi_3$ and $\xi_4 - \xi_3$respectively so that the summation is zero as shown in Fig. 3. We should point out that the choice of such complex weights is not unique.

Taking the linear constraint on every node, we derive a composite constraint for the framework as follows:$L\xi = 0$
Where is the complex Laplacian corresponding to the bidirectional graph *G* whose nonzero off-diagonal entry is $-w_{ij}$thenegative weight on edge (*j*, *i*). Now we are ready to introduce the notion of similar formation.

Definition 3.3: A framework (G,ξ)specified by $L\xi = 0$is *similar* if $\ker(L) = \{c_1 1_n + c_2\xi\colon\ c_1\, c_2 \in \mathbb{C}\}$

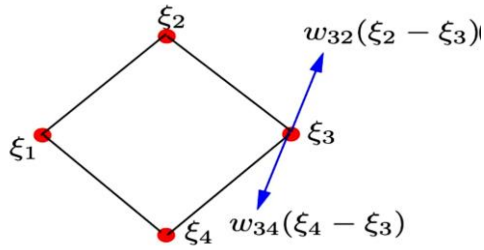

Fig. 3.Illustration of a linear constraint for a framework.

Remark 3.1: Note that a complex number $c_2$can be writtenin the polar coordinate form (namely $c_2 = \rho e^{i\theta}$).So the solutions to the linear constraint $L\xi = 0$consist of points relatedby translations c_1 rotations θ, and scaling ρ(four degrees offreedom). That is, the formations subject to the linear constraint $L\xi = 0$are scalable from the formation configuration ξinaddition to rigid body motions (translations and rotations). Therefore, one additional distance constraint on an edge will make the framework become globally rigid.

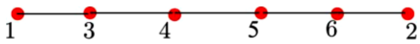

Fig. 4. A path graph of *n* nodes with its terminal nodes labelled as 1 and 2.

C. Necessary and Sufficient Conditions

In this subsection we are going to explore the necessary and sufficient algebraic and graphical conditions for similar frameworks.

Theorem 3.1: A framework (G,ξ)specified by $L\xi = 0$ is similar if and only if $rank(L) = n - 2$.
Proof: (Sufficiency) L has a zero eigenvalue with an associated eigenvector ξ because $L\xi = 0$Furthermore, since L is a Laplacian matrix, so $1_n = 0$, meaning that 1_nisanother eigenvector associated with the zero eigenvalue. Thetwo eigenvectors $\xi\ and 1_n$are linearly independent because $\xi_i \neq \xi_j$moreover, by the assumption $\mathrm{rank}(L) = n - 2$.we know that L has only two zero eigenvalues. Thus the null space of L is $c_1 1_n + c_2\xi\colon\ c_1\, c_2 \in \mathbb{C}$and so the framework (G,ξ)specified by $L\xi = 0$is similar (Necessity) suppose on the contrary that $rank(L) \neq n - 2$.
Then $rank(L)$must be less than $n - 2$since we already have $L\xi = 0$and$L1_n = 0$. Thus, it follows that the null space of L is of 3-dimension at least and $\ker(L) \neq \{c_1 1_n + c_2\xi\colon\ c_1\, c_2 \in \mathbb{C}\}$, which contradicts to the condition that the framework specified by $L\xi = 0$is similar.

Theorem 3.1 presents an algebraic condition requiring to check whether $rank(L) \neq n - 2$.
We give a graphical condition.

Theorem 3.2: A framework (G,ξ)specified by $L\xi = 0$(foralmost all L satisfying $L\xi = 0$is similar if and only if G is 2-rooted. The proof requires a lemma.

Lemma 3.1: Consider a framework $(G,\xi)$where G is a path graph of $n$nodes with its terminal nodes labeled as 1 and 2 (Fig. 4). If $\xi i \neq \xi j$for$i \neq j$,then there exists a complex Laplacian

$$L = \begin{bmatrix} A_{2\times 2} & B_{2\times(n-2)} \\ C_{(n-2)\times 2} & D_{(n-2)\times(n-2)} \end{bmatrix}$$

such that $L\xi = 0$and $D$is of rank$n - 2$.
Proof: If necessary, relabel the internal nodes of the path graph G in an order from 3 to $n$as shown in Fig. 4. Underthis labelling scheme, it is then clear that $D$is tri-

diagonal.Denote the first row of $D$by $d_1^T$and the remaining rows of D by $\bar{D}$Moreover, note that node 1 has only one neighbor (namely, node 3), so in the first column of Conly the (1,1)- entry is nonzero by the definition of LDenote the (1,1)-entry of $C$by c_1Then we can write C as

$$C=\begin{bmatrix} c_1 & 0 \\ 0 & \bar{c_2} \end{bmatrix}$$

Where $\bar{c_2} \in \mathbb{C}^{(n-3)}$From the definition of Laplacian, it followsthat$c_1 = -d_1^T\ 1, \bar{c_2} = -\bar{D}1$

Suppose for an$L$satisfying $L\xi = 0$ that $D$is not of rank $n - 2$.Moreover, notice that the rows of $\bar{D}$ are linearly independent.So there must exist an(n −3)dimensional vector λsuch that$d_1^T = \lambda^T \bar{D}$Moreover, using (1), we obtain that$c_1 = \lambda^T \bar{c_2} \neq 0$From$L\xi = 0$, thus we have

$$c_1\xi_1 + d_1^T\xi' = 0 \qquad (2)$$

and

$$\bar{c_2}\xi_2 + \bar{D}\xi' = 0 \quad (3)$$

Where $\xi'$is the sub-vector formed by the last n −2 entries of ξ.Pre-multiplying $\lambda^T$to (3) and using$c_1 = \lambda^T\bar{c_2}$ and $d_1^T = \lambda^T\bar{D}$ results in

$$c_1\xi_2 + d_1^T\xi' = 0 \qquad (4)$$

Comparing (4) and (2) we obtain that$\xi_1 = \xi_2$a contradiction.

Therefore, D is of rank $n - 2$.

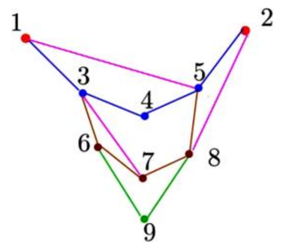

Fig. 5. Example of the relabeling procedure, where $U0 = \{1, 2\}$, $U1 =$
$\{3, 4, 5\}$, and so on.

Proof of Theorem 3.2: (Sufficiency) If G is 2-rooted, then from Definition 2.2, there is a subset of two nodes, from which every other node is 2-reachable. Without loss of generality, denote the subset by $U_0$and label the two nodes in U_0 by 1and 2. Select any node i not in U_0 and then we can find twodisjoint paths (no common nodes in these two paths except i)from 1 to i and from 2 to i since node i is 2-reachable fromU_0. Denote the set of nodes in these two paths excluding thenodes in U_0 by U_1 and denote n_1 the total number of nodes in U_1. Relabel the nodes in U_1 from 3 to n_1 + 2. The next stepis then to select another node, say j, not in $U_0 \cup U_1$. Also, because node is 2-reachable fromU_0, there must be twodisjoint paths from two different nodes in $U_0 \cup U_1$to node j, for which only the two terminal nodes are in $U_0 \cup U_1$. Denote$n_2$the total number of nodes in these two paths excluding the two terminal nodes in $U_0 \cup U_1$and relabel these nodes from $n_1 + 3$ to$n_1 + n_2 + 2$. Repeat the procedure until all the nodes are included. An illustration is presented in Fig. 5. According to the procedure, it is clear that$\sum_i n_i + 2 = n$
Take the graph $G'$with only edges included in the paths in the procedure. It is a subgraph of G with the same node set.

Notice that if a node i in U_{m1} is also a terminal node of some paths composed of nodes in U_{m2} for some $m_2 > m_1$, this node has more than two neighbours as it already has two neighbours in$U_{k=0,\dots,m1}$ $_{U_{m1}}U_k$

So we can select 0 for the complex weight $w_{i,j}$where$i \in U_{m1}$ and $j \in U_{m2}$ with $m_2 > m_1$

Thus, the Laplacian L' is of the following form:

$$L' = \begin{bmatrix} L_0 & * & * & * \\ * & L_0 & 0 & 0 \\ * & * & L_0 & 0 \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

where$L_i$is the corresponding block to the the subgraph induced by$U_i$in G' By our construction, we know that the subset $U_i$of nodes together with its two terminal nodes forma path graph. Thus, by applying Lemma 3.1 it follows thatrank(L_i) = n_i. Considering the particular structure of L' we know that

$$rank(L') \geq \sum_{i=1,\dots} rank(L_i) = \sum_{i=1,\dots} n_i = n - 2$$

Notice that L'can be considered as a Laplacian of the graph G for a special choice of weights with some being 0. Thus, by using the fact that either a polynomial is zero or it is not zero almost everywhere, it follows that for almost all complex weights satisfying$L\xi = 0$, there exists a non-zero principalminor of (n −2)th order. Therefore, $rank(L) \geq n - 2$. On the other hand, since $\xi$and $1_n$are two independent eigenvectorsof L corresponding to the zero eigenvalue, we have rank(L) =n −2. As a result of Theorem 3.1, the framework (G, ξ) specifiedby $L\xi = 0$(for almost all L satisfying $L\xi = 0$) is similar.(Necessity) We prove it in a contrapositive form. Supposethat the graph G is not 2-rooted. As a result, we cannot find two nodes to be roots from which all the nodes are 2-reachable.Since $L\xi = 0$ and $L\mathbf{1} = 0$, there must be two rows of L, say $l_q$and l_p, which can be transformed to zero vectors by elementaryrow operations. Choose the two nodes p and q corresponding to the two rows as roots and after removing a node, some node sare not reachable from the subset of roots. Without loss of generality, suppose after removing a node k there exist a subset W consisting of k −1 nodes which are not reachable from any root and a set$\bar{W}$consisting of $n - k$ nodes which are reachable from one of the roots. Relabel the nodes in W as 1, . . . , k -1and relabel the nodes in $\bar{W}$as k + 1, . . . , n. Then it is certain that the nodes in W are not reachable from any node in $\bar{W}$.Equivalently, $L(i, j)$ = 0 for $i \in W$ and $j \in \bar{W}$. Thus L is of the following form:

$$\begin{bmatrix} L_w & c_w & 0 \\ * & * & * \end{bmatrix}$$

Where $L_w \in \mathbb{C}^{(k-1)\times(k-1)}$ and $C_w \in \mathbb{C}^{(k-1))}$Denote the formationconfiguration ξafter relabelingby$[\xi_a^T, \xi_b^T]^T$ where

$\xi_a \in \mathbb{C}^k$ and $\xi_b \in \mathbb{C}^{k-2}$According to the definition of L, then we have $[L_w c_w]1_k = 0$ and $[L_w c_w]\xi_a = 0$ As $\mathbf{1}_k$and $\xi_a$are linearly independent by assumption, thenRank $([L_w c_w]) \leq k-2$. That is, there exists a row which canbe turned into the zero vector under elementary row operations.Therefore, rank(L) $\leq n-3$, or equivalently by Theorem 3.1, it is not true that the framework (G, ξ) specified by $L\xi = 0$ is similar.

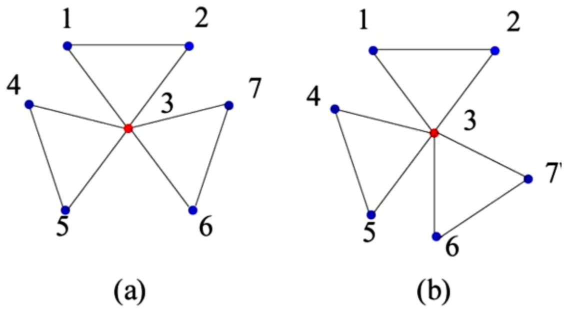

Fig. 6. If a graph G is not 2-rooted then the framework (G, ξ) specified by the distance constraint $g(\xi) = d$ is not rigid.

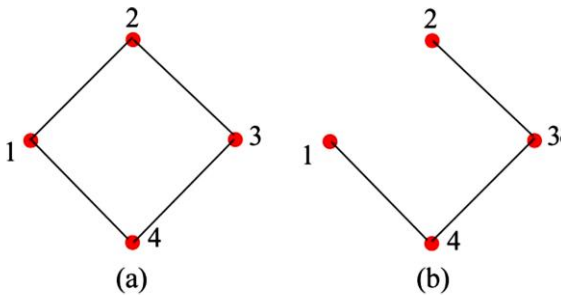

Fig. 7. Frameworks (G, ξ) specified by the distance constraint $g(\xi) = d$ are not rigid, but they are similar when specified by the linear constraint $L\xi = 0$.

Theorem 3.2 shows that 2-rooted connectivity is a necessary and sufficient condition for a framework (G, ξ) specified by the linear constraint $L\xi = 0$ to be similar for almost all complexLaplacian $L(G)$.

Remark 3.2: It is worth to point out that a graph G (of$n \geq 3$ nodes) for a rigid framework (G, ξ) using the distanceconstraint $g(\xi) = d$ must also be 2-rooted. This can be seen by the following fact. If G is not 2-rooted, then for any sub set of two nodes, there always exists another node that is not 2- reachable from the subset. That is, after removing a node, the graph can be divided into at least two sub graphs that are not connected to each other. An example is given in Fig. 6, for which after removing node 3, it results in three sub graphs that are not connected. This means, in addition to rigid body motions, another motion exists while preserving the distance constraint $g(\xi) = d$ [see for example Fig. 6(a) and 6(b)].However, the reverse is not true. In other words, to make a framework (G, ξ) specified by the distance constraint $g(\xi)$ =d rigid, the graph G requires more links than just 2-rootedconnectivity. From the well-known result by Lama in 1970[26], the minimal requirement for a framework specified by $g(\xi) = d$ to be rigid is that the graph should have at least $2n$ -3edges where n is the number of nodes. From our analysis we can know that the minimally 2-rooted graph requires only n -1edges, which corresponds to the path graph. So it requires much less links when specifying a similar framework in terms of the linear constraint $L\xi = 0$.In Fig. 7, both (a) and (b) are not rigid if the framework isspecified by the distance constraint $g(\xi) = d$, while they are similar if the framework is specified by the linear constraint$L\xi = 0$. Fig. 7(b) is a minimally 2-rooted graph that has only$n - 1$ edges.

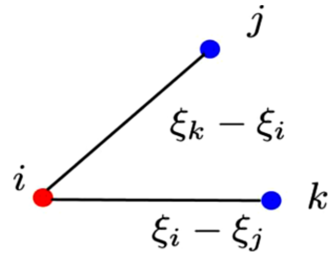

Fig. 8. Example of weight selection for a node having two neighbors

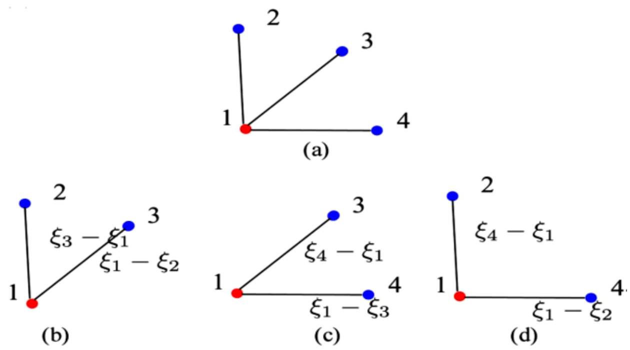

Fig. 9. Example of weight selection for a node having more than two neighbors

D. A Systematic Approach for the Construction of L

In the following, we present a systematic approach for the construction of L from the individual viewpoint. That is, for a given 2-rooted graph G and a formation configuration ξ, each agent i finds the weights wij's for $j \in Ni$ such that $L\xi = 0$.

As we discussed above, every node of a 2-rooted graph has at least two neighbors. In the following, we consider two cases. First, consider the case that node i has exactly two neighbors. Suppose without loss of generality, its two neighbors are j and k. Then the weights wij and wik can be parameterized as follows:

$$[wijwik] = p1i[\ \xi k - \xi i \xi i - \xi j\ ]$$

Where $p1i$is a nonzero complex number and can be chosen randomly. That is, [$wijwik$] is in the linear span of [$\xi k - \xi i \xi i - \xi j$]that solely depends on the formation configuration ξ. An example is given in Fig. 8.Second, consider the case that node $i$has more than two neighbors. Say without loss of generality that it has totally$m$ (m >2) neighbors, labelled by $i1$, . . . ,im. Select any two neighbors, denoted by *ij* and *ik*, from the *m* neighbors, and define an *m*-dimensional vector *ζh* with the *ij*th entry being*ξik*−*ξi*, the *ik*th entry being *ξi*−*ξij*, and the others being zero. Note that there are totally *C*2*m*(the binomial coefficient) selections of two neighbors out of *m*neighbors. Thus, theweights *wii*1, . . . ,*wiim*can be parameterized as follows:

[*wii*1···*wiim*] =*C*2*m*_*h*=1*phiζh* (5)where*phi* , *h* = 1, . . . ,

*C*2*m*, is a nonzero complex number and can be chosen randomly. An illustrative example is given Fig. 9(a) for which node 1 has three neighbors. So it has three choices of selecting any two neighbors as shown in Fig.9(b)-(d). Then the weight vector is a linear combination according to (5).

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